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The fractional finite Hankel transform and its applications in fractal space

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Abstract
In the present work, a generalized finite Hankel transform is derived which is useful in solving equations in fractal dimension d_f and involving a fractal diffusion coefficient $D_0 r^{-\theta}$. The corresponding inversion formula is established and some properties are given. Then, the transform is successfully used to solve a class of time-fractional diffusion equations in fractional spatial dimension with an absorbent term and Schrödinger equation in fractional-dimensional space. Green's functions and exact wave function of the above problems are found.

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1. Introduction

An integral transform method has been widely applied to the solution of boundary value problems of mathematical physics. The application of such transforms often reduces a partial differential equation to an ordinary differential equation [1–6]. The Hankel integral transform arises naturally in the discussion of problems posed in cylindrical coordinates and hence involves Bessel functions as a result of separation of variables. The method of the finite Hankel transform was first introduced by Sneddon [6]. Ali and Kalla [7] introduced a generalized form of the infinite Hankel transform and applied it to the problem of a heavy pollutant from a ground level aerial source within the framework of diffusion theory. Recently, Eldabe *et al* [8] gave another definition of the finite Hankel transform and applied it to the problem of unsteady flow through a concentric annulus. Many of these studies are concentrated on the d-dimensional ($d = 1, 2, 3$) diffusion equation with an absorbent term. Although the embedding space in our world is a three-dimensional (3D) Euclidean space, the motion of material objects is not always in three dimensions. At present, the time-fractional diffusion equation and the Schrödinger equation in fractional spatial dimension have not received much attention. It is the purpose of this paper to introduce a fractional finite Hankel transform, in which the finite

Hankel transform introduced by Sneddon [6] will emerge as a particular case, and to extend its utility to a wider class of time-fractional partial differential equations and the Schrödinger equation with fractional spatial dimension. This paper is organized as follows. In section 2, we show how a generalized finite Hankel transform and corresponding inversion formula can be obtained on fractal space which are referred to as the fractional finite Hankel transform. In section 3, firstly, the transform is successfully applied to solving a class of the time-fractional diffusion equation with an absorbent term which has a derivative of noninteger order. It can be proved that this kind of diffusion equation is very successful in describing anomalous kinetics, transport and chaos. We find Green's functions and exact solutions of every kind of boundary value problem. Secondly, the transform is applied to solving the Schrödinger equation in fractional-dimensional space. In section 4 we present our conclusions.

2. The fractional finite Hankel transform

A great variety of reactions in nature, the economy and engineering are heterogeneous. The geometrical structure of reaction surface takes on a complicated character [9] and has the property of a fractal structure. In recent years, the phenomenon of anomalous diffusion in fractal media has attracted more and more attention. Fractional anomalous diffusion equations have played an important part in describing anomalous diffusion phenomena in nature [10–14]. A large number of fractional diffusion equations taking place in the formulation of the problems of physics and engineering are specializations of the following general form:

$$\frac{1}{r^{d_f-1}}\frac{\partial}{\partial r}\left[r^{d_f-1}D_0r^{-\theta}\left(\frac{\partial P(r,t)}{\partial r}\right)\right] = {}_0D_t^{\nu}P(r,t)+\alpha(t)P(r,t), \tag{1}$$

with $0 < \nu \leqslant 1$. Here $D_0r^{-\theta}$ denotes the fractal diffusion coefficient, $\alpha(t)$ plays the role of an absorbent ($\alpha(t) < 0$) (or source ($\alpha(t) > 0$)) rate related to a reaction–diffusion process and ${}_0D_t^{\nu}$ is the Caputo fractional derivative of order ν defined as

$${}_0D_t^{\nu}f(t) = \frac{1}{\Gamma(n-\nu)}\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\nu+1-n}}\mathrm{d}\tau \qquad (n-1 < \mathrm{Re}(\nu) \leqslant n).$$

In this section we shall consider equation (1) together with the following homogeneous boundary condition to introduce a fractional finite Hankel transform:

$$\left(k\frac{\partial P(r,t)}{\partial r}+hP(r,t)\right)\Big|_{r=R} = 0. \tag{2}$$

The integral transform pair needed for the solution of the above problem can be developed by considering the following eigenvalue problem:

$$\frac{1}{r^{d_f-1}}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{\eta}\frac{\mathrm{d}\psi}{\mathrm{d}r}\right)+\lambda_m^2\psi = 0, \tag{3}$$

$$\left(\frac{\mathrm{d}\psi}{\mathrm{d}r}+h\psi\right)\Big|_{r=R} = 0, \qquad |\psi(\lambda_m,0)| < \infty, \tag{4}$$

where $\lambda_m (m = 1, 2, 3, \ldots)$ are eigenvalues and $\eta = d_f - 1 - \theta \geqslant 0$. The eigenfunctions $\psi(\lambda_m, r)$ (see equation (13)) of this eigenvalue problem are orthogonal with respect to the weighting function r^{d_f-1}, that is,

$$\int_0^R r^{d_f-1}\psi(\lambda_m,r)\psi(\lambda_n,r)\,\mathrm{d}r = \begin{cases}0, & \text{for} \quad m\neq n\\ N(\lambda_m), & \text{for} \quad m=n,\end{cases} \tag{5}$$

where the normalization integral $N(\lambda_m)$ is defined as

$$N(\lambda_m) = \int_0^R r^{d_f-1}(\psi(\lambda_m, r))^2\,\mathrm{d}r. \tag{6}$$

We now consider the representation of a function $P(r, t)$ defined in the finite region $(0, R)$ in terms of the eigenfunctions $\psi(\lambda_m, r)$ in the form

$$P(r, t) = \sum_{m=1}^{\infty} C_m(t)\psi(\lambda_m, r) \qquad \text{in } R, \tag{7}$$

where the summation is taken over the discrete spectrum of eigenvalues λ_m. In order to determine the unknown coefficients, multiplying $r^{d_f-1}\psi(\lambda_n, r)$ on both sides of equation (7) integrating from zero to R, and then using the orthogonality (5), we obtain

$$C_m(t) = \frac{1}{N(\lambda_m)}\int_0^R r^{d_f-1}\psi(\lambda_m, r)P(r, t)\,\mathrm{d}r. \tag{8}$$

Substituting equation (8) into equation (7) yields

$$P(r, t) = \sum_{m=1}^{\infty}\frac{\psi(\lambda_m, r)}{N(\lambda_m)}\int_0^R r'^{d_f-1}\psi(\lambda_m, r')P(r', t)\,\mathrm{d}r'. \tag{9}$$

Based on the above-mentioned analysis we introduce a fractional finite Hankel transform as follows.

Definition 1. *If $f(r)$ is continuously two-times differentiable on the interval $[0, R]$, then the fractional finite Hankel transform of order μ of a function $f(r)$ is denoted by $\Re_\mu\{f(r)\} = \widetilde{f}_\mu(\lambda_m)$ and is defined by*

$$\Re_\mu\{f(r)\} = \widetilde{f}_\mu(\lambda_m) = \int_0^R r^{d_f-1}f(r)\psi_\mu(\lambda_m, r)\,\mathrm{d}r. \tag{10}$$

The inverse finite Hankel transform is then defined by

$$\Re_\mu^{-1}\{\widetilde{f}_\mu(\lambda_m)\} = f(r) = \sum_{m=1}^{\infty}\frac{\psi_\mu(\lambda_m, r)}{N(\lambda_m)}\widetilde{f}_\mu(\lambda_m), \tag{11}$$

where λ_m are the positive roots of the equation

$$\lambda_m J'_\mu(\lambda_m bR^a) + hJ_\mu(\lambda_m bR^a) = 0, \tag{12}$$

and J_μ is the Bessel function of the first kind of order μ. The eigenfunctions of this eigenvalue problem $\psi(\lambda_m, r)$ are denoted by

$$\psi_\mu(\lambda_m, r) = \begin{cases} 1 & \text{for} \quad \lambda_m = 0 \\ r^{\frac{1-\eta}{2}}J_\mu(\lambda_m br^a) & \text{for} \quad \lambda_m > 0 \end{cases}, \tag{13}$$

and $\mu = 1 - \frac{d_f}{\theta+2}$, $a = \frac{2+\theta}{2}$, $b = \frac{2}{2+\theta}$.

Lemma 1. *If $f(r)$ is continuous on the interval $[0, R]$, then $\Re_\mu\{f(r)\} = \widetilde{f}_\mu(\lambda_m)$ exists, which converges absolutely and uniformly.*

Proof. It is clear that

$$\begin{aligned}|\widetilde{f}_\mu(\lambda_m)| &= \left|\int_0^R r^{d_f-1}f(r)\psi_\mu(\lambda_m, r)\,\mathrm{d}r\right| \leqslant \int_0^R \left|r^{\frac{d_f+\theta}{2}}f(r)J_\mu(\lambda_m br^a)\right|\mathrm{d}r \\ &= \int_0^R \left|r^{\frac{\eta}{2}+\frac{3\theta}{4}}f(r)b^{-1/2}\lambda_m^{-1/2}(\lambda_m br^a)^{\frac{1}{2}}J_\mu(\lambda_m br^a)\right|\mathrm{d}r.\end{aligned}$$

Since $f(r)$ is continuous on the interval $[0, R]$, $r^{\frac{\eta}{2}+\frac{3\theta}{4}} f(r)$ is continuous and bounded. When $\mu \geqslant -\frac{1}{2}$, the function $(\lambda_m b r^a)^{\frac{1}{2}} J_\mu(\lambda_m b r^a)$ is bounded. We have

$$|\widetilde{f}_\mu(\lambda_m)| \leqslant M_1 R \lambda_m^{-1/2} = M\lambda_m^{-1/2},$$

where M_1 and M are constant.

Hence, the integration

$$\widetilde{f}_\mu(\lambda_m) = \int_0^R r^{d_f-1} f(r)\psi_\mu(\lambda_m, r)\,\mathrm{d}r,$$

converges absolutely and uniformly. This proves the above lemma. □

Lemma 2. *Let $f(r)$ be continuously two-times differentiable on the interval* $[0, R]$ *and satisfy the boundary condition (4). Then the inverse formula (11) is uniformly convergent.*

Proof. The boundary value problems (3) and (4) are the singular Sturm–Liouville problem when $\eta = d_f - 1 - \theta > 0$ and are the regular Sturm–Liouville problem when $\eta = d_f - 1 - \theta = 0$. The endpoint 0 is singular endpoint [15, 16]. Equation (3) is in the limit-circle case at $r = 0$ if for some $\lambda \in C$ all solutions of equation (3) satisfy

$$\int_0^R r^{d_f-1}|\psi_\mu(\lambda_m, r)|^2\,\mathrm{d}r < +\infty \tag{14}$$

(see [15], p 277). So it suffices to consider equation (3) with $\lambda = 0$, which has two linearly independent solutions: $\psi_1(r) = 1$ and $\psi_2(r) = r^{-\eta+1}$ in the case of $\eta \neq 1$, $\psi_1(r) = 1$ and $\psi_2(r) = \ln r$ in the case of $\eta = 1$. It can be easily shown that ψ_1, ψ_2 satisfy equation (14) if and only if $0 < d_f < 4 + 2\theta$. Hence, equation (3) is in the limit-circle case at $r = 0$ when $-\frac{1}{2} \leqslant \mu < 1$. Based on the above discussion, it can be concluded that $r = 0$ is limit-circle non-oscillatory (LCNO) (quasi-regular) in the case of $1+\theta \leqslant d_f \leqslant 3+3\theta/2$ (see [15], p 278).

Now we introduce the inner product

$$\langle f, g\rangle = \int_0^R r^{d_f-1} f(r)g(r)\,\mathrm{d}r. \tag{15}$$

Equation (3) can be put in the usual eigenvalue problem form

$$L\psi(\lambda_m, r) = \lambda_m^2\psi(\lambda_m, r), \tag{16}$$

by defining a Sturm–Liouville operator $L = -\frac{1}{w(r)}\left(\frac{\mathrm{d}}{\mathrm{d}r}\left(p\frac{\mathrm{d}}{\mathrm{d}r}\right)+q\right)$, where $p = r^\eta$, $w(r) = r^{d_f-1}$ and $q = 0$. Let $f(r)$ and $g(r)$ be two functions having continuous second derivatives on the interval $[0, R]$ and satisfy the boundary condition (4); then

$$\langle Lf, g\rangle = -\int_0^R \frac{\mathrm{d}}{\mathrm{d}r}\left(p\frac{\mathrm{d}f}{\mathrm{d}r}\right) g\,\mathrm{d}r. \tag{17}$$

With integration by parts, equation (17) is reduced to

$$\langle Lf, g\rangle = \langle f, Lg\rangle - p(f'g - fg')\big|_0^R. \tag{18}$$

It is clear that if $p(f'g - fg')\big|_0^R = 0$ then $\langle Lf, g\rangle = \langle f, Lg\rangle$. By taking the boundary condition (4) and $p(0) = 0$, we obtain $p(f'g - fg')\big|_0^R = 0$. Hence, the operator L is self-adjoint under the boundary condition (4). If equations (3) and (4) are LCNO (quasi-regular) and L is self-adjoint, by using the results in [17], pp 110–25, and [18], pp 19–25, then L has a discrete spectrum and the eigenvalues are unbounded from above, that is $\lambda_n \to \infty, n \to \infty$. The conclusions of a self-adjoint operator eigenvalue problem remain valid under these above conditions (see [19], pp 284–6). So series (11) is uniformly convergent. Now we prove these conclusions. We use Green's function to arrive at an integral expression for L^{-1}, and define the operator T on $C([0, R])$ by [16]

$$Tu(r) = \langle G(r,\xi), u(\xi)\rangle = \int_0^R \xi^{d_f-1} G(r,\xi)u(\xi)\,\mathrm{d}\xi,$$

where G is Green's function and satisfies the boundary value problems (3) and (4):

$$G(r,\xi) = \begin{cases} \frac{1}{1-\eta}(R^{1-\eta} - r^{1-\eta}) + \frac{1}{R^\eta h}, & 0 \leqslant \xi \leqslant r \leqslant R, \\ \frac{1}{1-\eta}(R^{1-\eta} - \xi^{1-\eta}) + \frac{1}{R^\eta h}, & 0 \leqslant r \leqslant \xi \leqslant R. \end{cases} \tag{19}$$

When $\eta = 1, \theta = 0$, equations (3) and (4) are classical problems that have been discussed in [21]. In the case of $\eta \neq 1$, since $T(Lu) = u$, using equation (16), we have $T\psi_\mu(\lambda_m) = \mu_m \psi_\mu(\lambda_m)$, where $\mu_m = 1/\lambda_m^2$. Bessel's inequality, applied to G as a function of ξ, yields

$$\sum_{m=1}^{n} \frac{1}{N(\lambda_m)} \mu_m^2 |\psi_\mu(\lambda_m, r)|^2 \leqslant \int_0^R \xi^{d_f-1} |G(r,\xi)|^2\,\mathrm{d}\xi.$$

Putting (19) into the right-hand side of the above inequality and carrying out some calculations, we obtain that the right-hand side integration exists and is bounded, that is $\int_0^R \xi^{d_f-1}|G(r,\xi)|^2\,\mathrm{d}\xi \leqslant M$. For any $u \in C([0, R])$,

$$\|T_n u\| = \left\| Tu - \sum_{m=1}^{n-1} \frac{\mu_m \langle u, \psi_\mu(\lambda_m)\rangle \psi_\mu(\lambda_m)}{N(\lambda_m)} \right\| \leqslant |\mu_n| \|u\|$$
$$= \frac{1}{\lambda_n^2}\|u\| \to 0, \qquad n \to \infty. \tag{20}$$

If $n > k$, then

$$\left| \sum_{m=k}^{n} \frac{\mu_m \langle u, \psi_\mu(\lambda_m)\rangle \psi_\mu(\lambda_m)}{N(\lambda_m)} \right| = \left| T \sum_{m=k}^{n} \frac{\langle u, \psi_\mu(\lambda_m)\rangle \psi_\mu(\lambda_m)}{N(\lambda_m)} \right|$$
$$\leqslant \|T\| \left\| \sum_{m=k}^{n} \frac{\langle u, \psi_\mu(\lambda_m)\rangle \psi_\mu(\lambda_m)}{N(\lambda_m)} \right\| \leqslant M \left(\sum_{m=k}^{n} \left| \left\langle u, \frac{\psi_\mu(\lambda_m)}{\sqrt{N(\lambda_m)}} \right\rangle \right|^2 \right)^{1/2}.$$

Here we use $|Tu| \leqslant M||u||$. The right-hand side of this inequality tends to zero as $k, n \to \infty$; hence the series $\sum_{m=1}^{\infty} \frac{\mu_m \langle u, \psi_\mu(\lambda_m)\rangle \psi_\mu(\lambda_m)}{N(\lambda_m)}$ converges uniformly on $[0, R]$. Tu and (20) now imply

$$Tu = \sum_{m=1}^{\infty} \frac{\mu_m \langle u, \psi_\mu(\lambda_m)\rangle \psi_\mu(\lambda_m)}{N(\lambda_m)}, \qquad x \in [0, R]. \tag{21}$$

If $f(r)$ satisfies the conditions of lemma 2, then $u = Lf$ and $f = Tu$. Since L is a self-adjoint operator, we have

$$\mu_m \langle u, \psi_\mu(\lambda_m)\rangle = \langle u, \mu_m \psi_\mu(\lambda_m)\rangle = \langle Lf, T\psi_\mu(\lambda_m)\rangle = \langle f, LT\psi_\mu(\lambda_m)\rangle = \langle f, \psi_\mu(\lambda_m)\rangle.$$

Using equation (21), we obtain that $f(r) = \sum_{m=1}^{\infty} \frac{\langle f, \psi_\mu(\lambda_m)\rangle \psi_\mu(\lambda_m)}{N(\lambda_m)}$ is uniformly convergent. So series (11) is uniformly convergent. □

Theorem 1. *If $f(r)$ is defined in $0 \leqslant r \leqslant R$ and satisfies the conditions of lemma 2, then $f(r)$ can be represented by the Fourier–Bessel series as*

$$f(r) = 2r^{\frac{1-\eta}{2}} \sum_{m=1}^{\infty} \widetilde{f}_{\mu}(\lambda_m) \frac{J_{\mu}(\lambda_m b r^a)}{J_{\mu}^2(\lambda_m b R^a)} \frac{\lambda_m^2 b}{b^2 R^{2a}\left(h^2 + \lambda_m^2\right) - \mu^2}, \tag{22}$$

and the infinite series (14) is uniformly convergent.

Proof. By means of equation (6) and definition 1, the normalization integral $N(\lambda_m)$ can be written as

$$N(\lambda_m) = \int_0^R r^{d_f - 1}(\psi_{\mu}(\lambda_m, r))^2 \, \mathrm{d}r = \int_0^R r^{1+\theta}(J_{\mu}(\lambda_m b r^a))^2 \, \mathrm{d}r. \tag{23}$$

Introducing a variable $y = br^a$, equation (23) becomes

$$N(\lambda_m) = \frac{1}{b} \int_0^{bR^a} y(J_{\mu}(\lambda_m y))^2 \, \mathrm{d}y. \tag{24}$$

Substituting $\psi(\lambda_m, r) = r^{\frac{1-\eta}{2}} J_{\mu}(\lambda_m b r^a)$, $y = br^a$ into equation (3), after some algebraical calculations, we get a new equation for $J_{\mu}(\lambda_m y)$,

$$y^2 J_{\mu}''(\lambda_m y) + \frac{y}{\lambda_m} J_{\mu}'(\lambda_m y) + \left(y^2 - \frac{\mu^2}{\lambda_m^2}\right) J_{\mu}(\lambda_m y) = 0. \tag{25}$$

Multiplying $2J_{\mu}'(\lambda_m y)$ on both sides of the above equation, equation (25) can be rearranged in the following form:

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[y^2 J_{\mu}'^2 + \left(y^2 - \frac{\mu^2}{\lambda_m^2}\right) J_{\mu}^2\right] - 2y J_{\mu}^2 = 0. \tag{26}$$

Integration of equation (26) gives

$$\int_0^{bR^a} y(J_{\mu}(\lambda_m y))^2 \, \mathrm{d}y = \frac{(bR^a)^2}{2}\left[J_{\mu}'^2(\lambda_m b R^a) + \left(1 - \frac{\mu^2}{(\lambda_m b R^a)^2}\right) J_{\mu}^2(\lambda_m b R^a)\right]. \tag{27}$$

Using equation (12) we get

$$N(\lambda_m) = \frac{bR^{2a}}{2}\left[\frac{h^2}{\lambda_m^2} + \left(1 - \frac{\mu^2}{(\lambda_m b R^a)^2}\right)\right] J_{\mu}^2(\lambda_m b R^a). \tag{28}$$

Substituting the value of $N(\lambda_m)$ into equation (11) gives equation (22). It follows from lemmas 1 and 2 that the infinite series (14) is uniformly convergent.

When the boundary condition is the second kind (or the first kind), that is, $h = 0$ (or $h \to \infty$), then the eigenvalues λ_m are the positive roots of the equation

$$J_{\mu}'(\lambda_m b R^a) = 0, \tag{29}$$

or

$$J_{\mu}(\lambda_m b R^a) = 0, \tag{30}$$

respectively. The solutions of the problem are immediately obtained from equations (11) and (16) as

$$f(r) = \frac{d_f}{R^{d_f}} \widetilde{f}_0(\lambda_0) + \sum_{m=1}^{\infty} 2r^{\frac{1-\eta}{2}} \widetilde{f}_{\mu}(\lambda_m) \frac{J_{\mu}(\lambda_m b r^a)}{J_{\mu}^2(\lambda_m b R^a)} \frac{b\lambda_m^2}{\lambda_m^2 b^2 R^{2a} - \mu^2}, \tag{31}$$

or

$$f(r) = \frac{2r^{\frac{1-\eta}{2}}}{bR^{2a}} \sum_{m=1}^{\infty} \widetilde{f}_{\mu}(\lambda_m) \frac{J_{\mu}(\lambda_m b r^a)}{J_{\mu+1}^2(\lambda_m b R^a)}. \tag{32}$$

Here we use the recurrence relation $J'_\mu(\lambda_m bR^a) = J_{\mu-1}(\lambda_m bR^a) = -J_{\mu+1}(\lambda_m bR^a)$, due to the standard relation $zJ'_\mu + zJ_{\mu+1} = \mu J_\mu$. The first term in equation (31) corresponds to the eigenvalue $\lambda_0 = 0$. For this particular case, $\lambda_0 = 0$ is also an eigenvalue corresponding to the eigenfunction $\psi_0 = \text{constant} \neq 0$. The normalization integral $N(\lambda_0)$ takes the form $N(\lambda_0) = \frac{R^{d_f}}{d_f}$. □

Theorem 2. *If $f(r)$ is defined in $0 \leqslant r \leqslant R$ and $\Re_\mu\{f(r)\} = \widetilde{f}_\mu(\lambda_m)$, then*

$$\Re_\mu\left\{\frac{1}{r^{d_f-1}}\frac{\partial}{\partial r}\left(r^\eta\frac{\partial f(r)}{\partial r}\right)\right\} = R^\eta\psi_\mu\left[\frac{\partial f}{\partial r}+hf\right]_{r=R} - \lambda_m^2\widetilde{f}_\mu(\lambda_m). \tag{33}$$

Proof. By use of definition 1 and lemma 1, we have

$$\Re_\mu\left\{\frac{1}{r^{d_f-1}}\frac{\partial}{\partial r}\left(r^\eta\frac{\partial f(r)}{\partial r}\right)\right\} = \int_0^R\frac{\partial}{\partial r}\left(r^{d_f-1-\theta}\frac{\partial f(r)}{\partial r}\right)\psi_\mu(\lambda_m r)\,\mathrm{d}r.$$

Applying integration by parts to the right-hand side of the above formula, we obtain the result

$$r^\eta\frac{\partial f}{\partial r}\psi_\mu(\lambda_m r)\big|_0^R - \int_0^R r^\eta\frac{\partial f}{\partial r}\frac{\partial\psi_\mu}{\partial r}\,\mathrm{d}r = R^\eta\left(\psi_\mu\frac{\partial f}{\partial r} - f\frac{\partial\psi_\mu}{\partial r}\right)\big|_{r=R} + \int_0^R f\frac{\partial}{\partial r}\left(r^\eta\frac{\partial\psi_\mu}{\partial r}\right)\mathrm{d}r.$$

As ψ_μ is a solution of the eigenvalue problem equation (3) under the boundary condition equation (4), we get after a little simplification

$$\Re_\mu\left\{\frac{1}{r^{d_f-1}}\frac{\partial}{\partial r}\left(r^\eta\frac{\partial f(r)}{\partial r}\right)\right\} = R^\eta\psi_\mu\left[\frac{\partial f}{\partial r}+hf\right]_{r=R} - \lambda_m^2\widetilde{f}_\mu(\lambda_m).$$

This proves the theorem. □

Definition 1, theorem 1 and theorem 2 converge to the classical finite Hankel transform [1, 6] when $d_f = 2, \theta = 0$. So the classical finite Hankel transform is a special case of this transformation. Result equation (33) can be used to solve the initial boundary value problems on fractal space.

3. Applications of the fractional finite Hankel transform

In order to illustrate the advantages and accuracy of the above transform for solving the anomalous diffusion equation in fractional spatial dimension, we apply it to two different examples in this section. One is the fractional anomalous diffusion equation with a fractional oscillator term, and the other is the Schrödinger equation in fractal space.

3.1. Fractional anomalous diffusion equation with a fractional oscillator term

In this section, we present the use of the above integral-transform technique in the solution of fractional-dimensional, time-fractional-dependent, homogeneous boundary value problems in finite regions. Let us start to investigate equation (1) in the presence of a fractional oscillator term. Thus, we focus our attention on the following equation:

$$\frac{1}{r^{d_f-1}}\frac{\partial}{\partial r}\left[r^{d_f-1-\theta}\left(\frac{\partial P(r,t)}{\partial r}\right)\right] = {}_0D_t^\nu P(r,t) + \frac{\alpha}{\Gamma(\beta)}\int_0^t(t-t')^{\beta-1}P(r,t')\,\mathrm{d}t', \tag{34}$$

with $0 < \nu \leqslant 1, 0 < \beta \leqslant 1$. The initial and boundary conditions are as follows:

$$P(r,0) = P_0(r), \qquad 0 < r < R, \tag{35}$$

$$\frac{\partial P(r,t)}{\partial r} + hP(r,t) = 0, \qquad r = R, \tag{36}$$

where $P(0,t)$ is bounded. Applying the Laplace transform of the Caputo fractional derivative [20] to equation (34) and conditions (35), (36), we get

$$(s^{\nu} + \alpha s^{-\beta})\overline{P}(r,s) - s^{\nu-1}P(r,0) = \frac{1}{r^{d_f-1}}\frac{\partial}{\partial r}\left(r^{d_f-1-\theta}\frac{\partial \overline{P}(r,s)}{\partial r}\right), \tag{37}$$

$$\left(\frac{\partial \overline{P}(r,s)}{\partial r} + h\overline{P}(r,s)\right)\Big|_{r=R} = 0, \qquad t \geqslant 0, \tag{38}$$

where $\overline{P}(r,s)$ is the image function of $P(r,\ t)$ and s is the Laplace transform parameter. Application of the above fractional finite Hankel integral transform defined by equation (10) and theorem 2 to equation (37) gives

$$(s^{\nu} + \alpha s^{-\beta})\widetilde{\overline{P}}(\lambda_m,s) - s^{\nu-1}\widetilde{P}(\lambda_m,0) = R^{\eta}\varphi_{\mu}\left[\frac{\partial \overline{P}(\lambda_m,s)}{\partial r} + h\overline{P}(\lambda_m,s)\right]_{r=R} - \lambda_m^2\widetilde{\overline{P}}(\lambda_m,s).$$

From the boundary condition, we get

$$\widetilde{\overline{P}}(\lambda_m,s) = \frac{s^{\nu-1}}{s^{\nu} + \alpha s^{-\beta} + \lambda_m^2}\widetilde{P}(\lambda_m,0), \tag{39}$$

where $\widetilde{P}(\lambda_m,0)$ is the fractional finite Hankel transform of the initial condition. In order to get $\widetilde{P}(\lambda_m,t)$ and to avoid lengthy calculations of residues and contour integrals, we apply the discrete inverse Laplace transform method to equation (39):

$$\begin{aligned}\widetilde{P}(\lambda_m,t) &= \mathcal{L}^{-1}\left[\frac{s^{\nu-1}\widetilde{P}(\lambda_m,0)}{s^{\nu} + \alpha s^{-\beta} + \lambda_m^2}\right] = \mathcal{L}^{-1}\left[\sum_{k=0}^{\infty}(-1)^k\frac{\alpha^k s^{\nu-1-k\beta}\widetilde{P}(\lambda_m,0)}{\left(s^{\nu}+\lambda_m^2\right)^{k+1}}\right]\\ &= \sum_{k=0}^{\infty}\frac{(-1)^k\alpha^k}{k!}t^{k\nu+k\beta}E^{(k)}_{\nu,1+k\beta}\left[-\lambda_m^2t^{\nu}\right]\widetilde{P}(\lambda_m,0).\end{aligned} \tag{40}$$

Here we used an important formula of the Laplace transform of the generalized Mittag–Leffler function:

$$\mathcal{L}^{-1}\left\{\frac{n!s^{\lambda-\mu}}{(s^{\lambda}\mp c)^{n+1}};t\right\} = t^{n\lambda+\mu-1}E^{(n)}_{\lambda,\mu}(\pm ct^{\lambda}),$$

in which $E_{\lambda,\mu}(z)$ is the generalized Mittag–Leffler function defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)} \qquad (\alpha > 0,\ \beta > 0),$$

and

$$E^{(k)}_{\alpha,\beta}(z) = \frac{d^k}{dz^k}E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{(n+k)!z^n}{n!\Gamma(\alpha n+\beta k+\beta)}.$$

Making use of the inverse fractional finite Hankel transform equations (22) to (40), we obtain the expression of Green's function in the form of

$$P(r,t) = \int_0^R r'^{d_f-1}G(r,t|r',\tau)|_{\tau=0}P_0(r')\,dr', \tag{41}$$

and

$$G(r,t|r',\tau)|_{\tau=0} = 2(rr')^{\frac{1-\eta}{2}} \sum_{m=1}^{\infty} \frac{J_\mu(\lambda_m b r^a)}{J_\mu^2(\lambda_m b R^a)} \frac{\lambda_m^2 b}{b^2 R^{2a}\left(h^2+\lambda_m^2\right)-\mu^2} \times \sum_{k=0}^{\infty} \frac{(-1)^k \alpha^k}{k!} t^{k\nu+k\beta} E_{\nu,1+k\beta}^{(k)}\left[-\lambda_m^2 t^\nu\right] J_\mu(\lambda_m b r'^a), \tag{42}$$

where $G(r,t|r',\tau)$ is Green's function. r'^{d_f-1} is the weighting function. Finally, replacing t by $t-\tau$, Green's function is given by

$$G(r,t|r',\tau) = 2(rr')^{\frac{1-\eta}{2}} \sum_{m=1}^{\infty} \frac{J_\mu(\lambda_m b r^a)}{J_\mu^2(\lambda_m b R^a)} \frac{\lambda_m^2 b}{b^2 R^{2a}\left(h^2+\lambda_m^2\right)-\mu^2} \times \sum_{k=0}^{\infty} \frac{(-1)^k \alpha^k}{k!} (t-\tau)^{k\nu+k\beta} E_{\nu,1+k\beta}^{(k)}\left[-\lambda_m^2 (t-\tau)^\nu\right] J_\mu(\lambda_m b r'^a). \tag{43}$$

When the boundary condition is the second kind (or the first kind), that is $h=0$ (or $h\to\infty$) and the eigenvalues λ_m are the positive roots of the equation $J'_\mu(\lambda_m b R^a)=0$ (or $J_\mu(\lambda_m b R^a)=0$), in the same way as deriving equation (42), we obtain

$$G(r,t|r',\tau) = \sum_{m=1}^{\infty} 2(rr')^{\frac{1-\eta}{2}} \frac{J_\mu(\lambda_m b r^a)}{J_\mu^2(\lambda_m b R^a)} \frac{b\lambda_m^2}{\lambda_m^2 b^2 R^{2a}-\mu^2} \times \sum_{k=0}^{\infty} \frac{(-1)^k \alpha^k}{k!} (t-\tau)^{k\nu+k\beta} E_{\nu,1+k\beta}^{(k)}\left[-\lambda_m^2 (t-\tau)^\nu\right] J_\mu(\lambda_m b r'^a), \tag{44}$$

or

$$G(r,t|r',\tau)|_{\tau=0} = \frac{2(rr')^{\frac{1-\eta}{2}}}{bR^{2a}} \sum_{m=1}^{\infty} \frac{J_\mu(\lambda_m b r^a)}{J_{\mu+1}^2(\lambda_m b R^a)} \times \sum_{k=0}^{\infty} \frac{(-1)^k \alpha^k}{k!} (t-\tau)^{k\nu+k\beta} E_{\nu,1+k\beta}^{(k)}\left[-\lambda_m^2 (t-\tau)^\nu\right] J_\mu(\lambda_m b r'^a). \tag{45}$$

In view of the fact that [20] $E_{1,1}(z)=\mathrm{e}^z$, under the conditions of $\nu=1, d_f=2, \theta=0$ and $\alpha=0$, equation (45) recovers the result of Green's function of the usual diffusion equation found in [21, 22] which is given by

$$G(r,t|r',\tau)|_{\tau=0} = \frac{2}{R^2} \sum_{m=1}^{\infty} \frac{J_0(\lambda_m r)}{J_1^2(\lambda_m R)} \mathrm{e}^{-\lambda_m^2 t} J_0(\lambda_m r'). \tag{46}$$

Therefore, Green's functions of every kind of boundary value problems of normal diffusion can be regarded as particular cases of this paper. By using the expressions of equations (41) and (42), we may find the exact solution when the initial condition satisfies $P(r,0)=\delta(r-r_0)$ as follows:

$$\begin{aligned} P(r,t) &= 2(r)^{\frac{1-\eta}{2}} \sum_{m=1}^{\infty} \frac{J_\mu(\lambda_m b r^a)}{J_\mu^2(\lambda_m b R^a)} \frac{\lambda_m^2 b}{b^2 R^{2a}\left(h^2+\lambda_m^2\right)-\mu^2} \\ &\quad \times \sum_{k=0}^{\infty} \frac{(-\alpha t^{\mu+\beta})^k}{k!} E_{\nu,1+k\beta}^{(k)}\left[-\lambda_m^2 t^\nu\right] \int_0^R r'^{\frac{1-\eta}{2}} r'^{d_f-1} J_\mu(\lambda_m b r'^a) \delta(r'-r_0)\,\mathrm{d}r' \\ &= \sum_{m=1}^{\infty} \frac{2J_\mu(\lambda_m b r^a)}{J_\mu^2(\lambda_m b R^a)} \frac{\lambda_m^2 b r^{\frac{1-\eta}{2}} r_0^{\frac{d_f+\theta}{2}} J_\mu(\lambda_m b r_0^a)}{b^2 R^{2a}\left(h^2+\lambda_m^2\right)-\mu^2} \sum_{k=0}^{\infty} \frac{(-\alpha t^{\mu+\beta})^k}{k!} E_{\nu,1+k\beta}^{(k)}\left[-\lambda_m^2 t^\nu\right], \end{aligned} \tag{47}$$

where δ denotes the Dirac-delta function.

3.2. The Schrödinger equation in fractal space

Zeilinger and Svozil [23] noted that the current discrepancy between the theoretical and experimental values of the anomalous magnetic moment of the electron could be resolved if the dimensionality of the space α is $\alpha = 3 - (5.3 \pm 2.5) \times 10^{-7}$. In a fractional-dimensional model, one can study the anisotropic excitation dynamics by solving the relevant Schrödinger equation in a noninteger-dimensional space. Eid *et al* [24] investigated the Schrödinger equation in a given α-dimensional fractional space with a Coulomb potential depending on a parameter. In this section the main aim is to apply the above integral-transform technique to solving the Schrödinger equation in a given d_f-dimensional fractional space with a spherical potential well, which takes the form of

$$\left[\frac{-\hbar^2}{2\upsilon r^{d_f-1}}\frac{\partial}{\partial r}\left(r^{d_f-1}\frac{\partial}{\partial r}\right)+V(r)\right]\chi(r,\theta,\varphi)=E\chi(r,\theta,\varphi). \tag{48}$$

The common eigenfunctions are

$$\chi(r,\theta,\varphi)=P(r)Y_{lm}(\theta,\varphi), \tag{49}$$

where $V(r)$ is the potential function, d_f is the dimension of a solid ($1 \leqslant d_f \leqslant 3$) and $Y_{lm}(\theta,\varphi)$ denote the harmonic spherical functions. We arrive at the radial equation in the form of

$$\frac{1}{r^{d_f-1}}\frac{\partial}{\partial r}\left(r^{d_f-1}\frac{\partial P(r)}{\partial r}\right)+\frac{2\upsilon}{\hbar^2}\left\{E-\left[V(r)+\frac{\hbar^2}{2\upsilon}\frac{l(l+1)}{r^2}\right]\right\}P(r)=0, \tag{50}$$

where l is an angular-momentum quantum number. When the potential function is a spherical potential well $V(r)=-\gamma\delta\left(r-\frac{R}{2}\right)$ ($\gamma > 0, 0 \leqslant r \leqslant R$) and bounded states are base states ($E < 0, l = 0$), the radial equation reads

$$\frac{1}{r^{d_f-1}}\frac{\partial}{\partial r}\left(r^{d_f-1}\frac{\partial P(r)}{\partial r}\right)+\frac{2\upsilon}{\hbar^2}\left\{E+\gamma\delta\left(r-\frac{R}{2}\right)\right\}P(r)=0. \tag{51}$$

The boundary conditions of wave function satisfy $P(0) = P(R) = 0$. With the help of the fractional finite Hankel integral transform defined by equation (10) and theorem 2, equation (51) becomes

$$R^\eta\left(\psi_\mu\frac{\partial P}{\partial r}-P\frac{\partial\psi_\mu}{\partial r}\right)\Big|_{r=R}-\lambda_m^2\widetilde{P}+\frac{2\upsilon}{\hbar^2}E\widetilde{P}+\frac{2\upsilon\gamma}{\hbar^2}\left(\frac{R}{2}\right)^{\frac{d_f}{2}}P\left(\frac{R}{2}\right)J_\mu\left(\frac{\lambda_m R}{2}\right)=0, \tag{52}$$

where we used the formula of the Dirac-delta function:

$$\int_0^R f(x)\delta(x-x_0)\,\mathrm{d}x=f(x_0), \tag{53}$$

and $\theta = 0$ (or $\eta = d_f - 1$) is used. Taking the boundary condition and $J_\mu(\lambda_m R) = 0$, we get

$$\widetilde{P}=\frac{\frac{2\upsilon\gamma}{\hbar^2}\left(\frac{R}{2}\right)^{\frac{d_f}{2}}P\left(\frac{R}{2}\right)J_\mu\left(\frac{\lambda_m R}{2}\right)}{\lambda_m^2-\frac{2\upsilon}{\hbar^2}E}. \tag{54}$$

Substituting equation (54) into equation (32), the wavefunction in a given d_f-dimensional fractional space with a spherical potential well is obtained:

$$P(r)=\frac{2r^{1-\frac{d_f}{2}}}{R^2}\sum_{m=1}^{\infty}\frac{J_\mu(\lambda_m r)}{J_{\mu+1}^2(\lambda_m R)}\frac{\frac{2\upsilon\gamma}{\hbar^2}\left(\frac{R}{2}\right)^{\frac{d_f}{2}}P\left(\frac{R}{2}\right)J_\mu\left(\frac{\lambda_m R}{2}\right)}{\lambda_m^2-\frac{2\upsilon}{\hbar^2}E}. \tag{55}$$

4. Conclusions

In summary, in this paper we introduce a fractional finite Hankel integral transform. Inversion formula is established and some properties are given. The transform is successfully applied to solving a class of the time-fractional diffusion equation with a fractional oscillator term. We find Green's functions and exact solutions of every kind of boundary value problem. At the same time, we use the transform to solve the Schrödinger equation in a given d_f-dimensional fractional space with a spherical potential well.

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